COHOMOLOGY THEORY OF ABELIAN GROUPS

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This note will present certain algebraic results obtained by Samuel Eilenberg and the author in a study of the relations between homotopy and homology groups of a topological space.[2] These results yield a homology theory for any abelian group Π, in which the low dimensional homology and cohomology groups of Π correspond to familiar constructions on Π. They depend upon the application of the methods of algebraic topology to algebraic systems. Some of the topological aspects of these constructions are presented by Eilenberg in another note in these Proceedings.

An abstract cell complex K is a sequence

$$(1) \qquad C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} C_3 \cdots C_{n-1} \xleftarrow{\partial_n} C_n \cdots$$

of abelian groups C_n and homomorphisms ∂_n such that the composition $\partial_n \partial_{n+1}: C_{n+1} \to C_{n-1}$ of two successive homomorphisms is the zero homomorphism; furthermore each group C_n is a free abelian group with a specified set of free generators σ, called the n-dimensional cells of K. The group Z_n of n-dimensional cycles of K is the kernel of ∂_n in C_n, for $n > 1$, and is C_1 for $n = 1$. The group B_n of n-dimensional boundaries is the image of ∂_{n+1} in C_n. Since $\partial_n \partial_{n+1} = 0$, $B_n \subset Z_n \subset C_n$. The n-dimensional integral homology group $H_n = H_n(K)$ is defined as Z_n/B_n.

The cohomology groups of the complex K may be defined for any abelian "coefficient group" G. The group $C^n(K; G)$ of n-dimensional cochains of K is the group of all homomorphisms $f: C_n \to G$, or equivalently the group of all functions f on the n-cells of K to G. The coboundary $\delta_n f$ is an $(n + 1)$-cochain defined as the composite homomorphism $f\partial_{n+1}: C_{n+1} \to G$. These definitions yield a sequence of groups and homomorphisms

$$C^1(K; G) \xrightarrow{\delta_1} C^2(K; G) \xrightarrow{\delta_2} C^3(K; G) \longrightarrow \cdots$$

with $\delta_{n+1}\delta_n = 0$. As before, one defines the cohomology group $H^n(K, G)$ as Z^n/B^n, where Z^n is the kernel of δ_n, B^n the image of δ_{n-1} for $n > 1$, and $B^1 = 0$.

Any group Q (not necessarily abelian) has a standard homology theory which is the homology theory of the cell complex $A^0(Q)$ constructed as follows. The n-cells of $A^0(Q)$ are all n-tuples $[x_1, \cdots, x_n]$ of elements x_i of Q, and the boundary homomorphisms ∂_n (we omit the subscript n) are obtained by setting

$$(2) \qquad \partial[x, y] = [y] - [xy] + [x],$$

$$(3) \qquad \partial[x, y, z] = [y, z] - [xy, z] + [x, yz] - [x, y],$$

[1] These investigations were started while the author held a John Simon Guggenheim Memorial Fellowship.

[2] S. Eilenberg and S. MacLane, *Cohomology theory of abelian groups and homotopy theory I and* II, Proc. Nat. Acad. Sci. U. S. A. vol. 36 (1950) pp. 443–447, 657–663.

and, more generally,

$$\partial[x_1, \cdots, x_n] = [x_2, \cdots, x_n] + \sum_{i=1}^{n-1} (-1)^i [x_1, \cdots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \cdots, x_n] + (-1)^n [x_1, \cdots, x_{n-1}], \tag{4}$$

in agreement with (2) and (3) for $n = 2$ and $n = 3$. The verification that $\partial\partial = 0$ depends only on the associative law for Q.

The groups of the complex $A^0(Q)$ are known as the homology and cohomology groups of Q. For example, the group Z_1 of one-dimensional cycles is the free abelian group with generators $[x]$, for $x \in Q$. If $[Q, Q]$ is the commutator group of Q, the homomorphism of Z_1 into $Q/[Q, Q]$ given by mapping each generator $[x]$ into the coset $x[Q, Q]$ has as kernel the group B_1 of one-dimensional boundaries $[y] - [xy] + [x]$. Hence the isomorphism

$$H_1(A^0(Q)) \cong Q/[Q, Q], \qquad \text{under } [x] \rightarrow x[Q, Q]. \tag{5}$$

A one-dimensional cochain is a function $f(x)$ defined on the 1-cells $[x]$ with values in G. It is a cocycle if $\delta f = 0$; by (2) this means that $f(y) - f(xy) + f(x) = 0$; i.e., that f is a homomorphism. Hence

$$H^1(A^0(Q), G) \cong \text{Hom}\,(Q, G), \tag{6}$$

the group of all homomorphisms of Q into G. Similarly, a two-dimensional cocycle f is a function $f(x, y)$ on 2-cells $[x, y]$ with values in G which satisfies, according to (3), the functional equation

$$f(y, z) + f(x, yz) = f(xy, z) + f(x, y). \tag{7}$$

Any such function is a "factor set" of Q in G. Each factor set determines a central group extension E of G by Q; specifically, E is the group of all pairs (x, g), for $x \in X$, $g \in G$, with the composition

$$(x, g)(y, h) = (xy, ghf(x, y)) \tag{8}$$

and the homomorphism $(x, g) \rightarrow x$ onto Q. The equation (7) insures that the product is associative. This extension E is equivalent to the direct product extension $Q \times G$ if and only if the cocycle f is a coboundary. In this fashion one may prove that

$$H^2(A^0(Q), G) \cong \text{Extcent}\,(Q, G),$$

where "Extcent" denotes the group of all central extensions of G by Q.

Instead of using explicit boundary formulas such as (4) we may characterize the cohomology groups of the complex $A^0(Q)$ by certain formal properties, using the special case when Q is the free group F with a fixed denumerable set of generators $g_1, g_2, \cdots$. In the complex $A^0(F)$ a cell $[x_1, \cdots, x_n]$ is called *generic* if each x_i is a product of zero or more generators, such that any one generator

g_j appears in at most one of these products x_i. Inspection of the boundary formula (4) shows that the boundary of a generic cell is a linear combination of generic cells. Hence the generic cells alone span a subcomplex $A^0(F^*)$ of $A^0(F)$. This "generic subcomplex" has the homology groups

$$H_n(A^0(F^*)) = 0, \qquad n > 1. \tag{9}$$

This property, together with (5), can be used to characterize the homology and cohomology groups of any group Q without reference to the specific complex $A^0(Q)$. It gives implicitly a rule for the construction of suitable complexes like A^0: given the cells through dimension n, enough cells in dimension $n + 1$ must be added to make every n-dimensional generic cycle a boundary.

There are several indications that the homology theory appropriate to an abelian group Π will not be given by the complex $A^0(\Pi)$. In the appropriate complex, the proof that $\partial\partial = 0$ should use both the associative and commutative laws valid in Π. In dimension 2, the cohomology groups of Π should correspond to extensions of G by Π which are abelian. For Q abelian, the extension E described in (8) will be abelian if the factor set f satisfies the additional condition

$$f(x, y) = f(y, x); \tag{10}$$

this indicates that two-dimensional "abelian" cocycles f should satisfy both (7) and (10).

We thus propose to construct complexes $A(\Pi)$ for additive abelian groups Π in such fashion that the generic subcomplex $A^0(F_a^*)$ for a free abelian group F_a will have vanishing higher homology groups, as in (9). The complex $A^0(\Pi)$ itself does not enjoy this property; indeed, if g and h are distinct generators of F_a, the 2-chain $[g, h] - [h, g]$ has boundary zero, hence is a generic cycle but not a generic boundary. We therefore adjoin to $A^0(\Pi)$ a new 3-cell $[x \mid y]$ with boundary

$$\partial[x \mid y] = [x, y] - [y, x]. \tag{11}$$

After this adjunction, the generic cycle $[g, h] - [h, g]$ becomes a boundary and the 2-dimensional cocycles $f(x, y)$ must satisfy not only the "associativity" condition (7), but also the "commutativity" condition (10). In dimension 4, we adjoin two more types of four cells $[x, y \mid z]$ and $[x \mid y, z]$, with boundaries

$$\partial[x, y \mid z] = [y \mid z] - [x + y \mid z] + [x \mid z] - [x, y, z] + [x, z, y] - [z, x, y], \tag{12}$$

$$\partial[x \mid y, z] = [x \mid z] - [x \mid y + z] + [x \mid z] + [x, y, z] - [y, x, z] + [y, z, x]. \tag{13}$$

Indeed, the expressions on the right in (12) and (13) would otherwise be nonbounding generic 3-cycles when x, y, z are distinct generators of F_a.

These formulas may be written more conveniently if we define a "shuffle" of m letters $x_1, \cdots, x_m$ through n letters $y_1, \cdots, y_n$ to be any list of these $m + n$ letters in an order which preserves both the order of the x's alone and that of the y's alone. The sign of the shuffle is the sign of the permutation required to bring the shuffled letters back to the standard order $x_1, \cdots, x_m, y_1, \cdots, y_n$. Finally, the "star" product $[x_1, \cdots, x_m]*[y_1, \cdots, y_n]$ is defined as the signed sum of all shuffles of the letters x through the letters y. The boundary formula (12) becomes, in this notation,

$$\partial[x, y \mid z] = [\partial(x, y) \mid z] - (x, y) * z,$$

with similar expressions for (13).

Generalizing the formulas (12) and (13), we construct the complex $A^1(\Pi)$ in which the cells are symbols $\sigma = [\alpha_1 \mid \cdots \mid \alpha_p]$, with each α_i a cell of $A^0(\Pi)$. The dimension of σ is $p - 1$ plus the sum of the dimensions of the α_i, and the boundary of σ is

$$\partial\sigma = \sum_{i=1}^{p} (-1)^{\epsilon_{i-1}} [\alpha_1 \mid \cdots \mid \partial\alpha_i \mid \cdots \alpha_p] + \sum_{i=1}^{p-1} (-1)^{\epsilon_i} [\alpha_1 \mid \cdots \mid \alpha_i * \alpha_{i+1} \mid \cdots \mid \alpha_p], \tag{14}$$

where $\epsilon_i = 1 + \dim [\alpha_1 \mid \cdots \mid \alpha_i]$. The proof that $\partial\partial = 0$ uses both the associative and the commutative laws in Π.

In the so constructed complexes $A^1(\Pi)$ all the generic cycles of $A^0(\Pi)$ become boundaries, but there are new generic cycles, such as the cycle $[g \mid h] + [h \mid g]$ of dimension 3. To kill this cycle, we add a new 4-cell $[x \,||\, y]$ with boundary

$$[x \,||\, y] = -[x \mid y] - [y \mid x]. \tag{15}$$

This is the first step in the construction of a new complex $A^2(\Pi)$ which has cells $[\sigma_1 \,||\, \cdots \,||\, \sigma_q]$, with the σ_i cells of $A^1(\Pi)$, and a boundary formula resembling (14) (but with a new shuffling operation which shuffles only the "blocks" α_j of each σ_i). The same construction[2] is then repeated inductively to give complexes $A^k(\Pi)$. The final complex $A(\Pi)$ is the union of all the complexes $A^k(\Pi)$; for a fixed dimension q the cells of $A(\Pi)$ are simply those of A^k with $k = q - 2$.

In the complex $A(\Pi)$ we may again define for the free abelian group F_a the generic subcomplex $A(F_a^*)$.

THEOREM 1. *For the construction A the generic subcomplex has the homology groups*

$$H_1(A(F_a^*)) \cong F_a, \qquad H_n(A(F_a^*)) = 0, \qquad n > 0. \tag{16}$$

There are many alternative constructions K of complexes $K(\Pi)$, one for each abelian group Π, which have as cells various types of n-tuples of elements of Π, with a boundary "formula" valid for all groups, and such that the generic cells of $K(F_a)$ form a subcomplex $K(F_a^*)$. We require also that the 1-cells of $K(\Pi)$ be the 1-cells $[x]$ of $A(\Pi)$.

THEOREM 2. *Let K be any such construction which has generic homology groups as in* (15). *Then for any abelian group* Π, *the homology and cohomology groups of* $K(\Pi)$ *in dimension n are isomorphic to those of* $A(\Pi)$.

We are thus justified in referring to the cohomology groups of $A(\Pi)$ as *the* cohomology groups of the *abelian* group Π.

These groups may be explicitly computed for any given dimension if Π is a finitely generated group. For certain low dimensions they have been determined for any Π. Thus, as in (6),

$$H^1(A(\Pi), G) = H^1(A^0(\Pi), G) \cong \text{Hom}(\Pi, G),$$

while, as already indicated in (10) and (11)

$$H^2(A(\Pi), G) = H^2(A^1(\Pi), G) \cong \text{Ext}(\Pi, G),$$

the group of all abelian group extensions of G by Π.

A 3-dimensional cochain of $A^1(\Pi)$ is a pair of functions $f(x, y, z) \in G$, $d(x \mid y) \in G$; they form a cocycle if they vanish on the boundaries (10), (11), and (4); that is, if they satisfy the identities

$$(17)\quad d(x + y \mid z) - d(y \mid z) - d(x \mid z) + f(x, y, z) - f(x, z, y) + f(z, x, y) = 0$$

$$(18)\quad d(x \mid y + z) - d(x \mid y) - d(x \mid z) - f(x, y, z) + f(y, x, z) - f(y, z, x) = 0$$

$$(19)\quad f(y, z, t) - f(x + y, z, t) + f(x, y + z, t) + f(x, y, z + t) - f(x, y, z) = 0.$$

A two-dimensional cochain is a function $h(x, y) \in G$; its coboundary is the pair (f, d) with

$$(20)\qquad f(x, y, z) = h(y, z) - h(x + y, z) + h(x, y + z) - h(x, y),$$

$$(21)\qquad d(x \mid y) = h(x, y) - h(y, x).$$

To any cocycle (f, d) we assign the function $t(x) = d(x \mid x) \in G$ as its trace. By (21), the trace of a coboundary is zero, and one may show that any trace satisfies the identities

$$(22)\quad t(x + y + z) - t(x + y) - t(y + z) - t(z + x) + t(x) + t(y) + t(z) = 0,$$

$$(23)\qquad t(x) = t(-x).$$

These are incidentally the formal identities satisfied by a "square" $t(x) = x^2$; hence we call any such function t a quadratic function.

THEOREM 3. *The function assigning to each cocycle its trace induces an isomorphism of* $H^3(A^1(\Pi), G)$ *to the group of all quadratic functions on* Π *to* G.

In the complex $A(\Pi)$ the three-cochains are the same pairs (f, d), but a cocycle must satisfy one additional identity $d(x \mid y) + d(y \mid x) = 0$, derived from (15). Thus $2t(x) = 0$, and we have

$$H^3(A(\Pi), G) = H^3(A^2(\Pi), G) \cong \text{Hom}(\Pi, {}_2G),$$

where ${}_2G$ is the subgroup of all elements of order 2 in G.

There are parallel results for the homology groups; one has the isomorphisms

$$H_1(A(\Pi)) \cong \Pi, \qquad H_2(A(\Pi)) = 0, \qquad H_3(A(\Pi)) \cong \Pi/2\Pi,$$

while $H_3(A^1(\Pi))$ is the group[3] $\Gamma(\Pi)$ which has the generators $[x]$ for $x \in \Pi$ and the relations

$$[x + y + z] - [x + y] - [y + z] + [z + x] + [x] + [y] + [z] = 0,$$
$$[x] = [-x].$$

In dimension 4, $H_4(A^1(\Pi))$ is isomorphic to the abelian group with the following generators: a generator $[s]$ for each element s of ${}_2\Pi$ and a generator $[x, y; h]$ for each pair of elements $x, y \in \Pi$ and each integer h such that $hx = hy = 0$. These generators satisfy the following relations for x, y, z with $hx = hy = hz = 0$

$$[x, y + z; h] = [x, y; h] + [x, z; h], \tag{24}$$

$$[x + y, z; h] = [x, z; h] + [y, z; h], \tag{25}$$

$$[x, x; h] = 0, \tag{26}$$

and the relations

$$[x, y; kh] = [kx, y; h], \quad \text{if} \quad (kh)x = 0, \qquad hy = 0, \tag{27}$$

$$[x, y; 2] = [x + y] - [x] - [y], \quad \text{if} \quad 2x = 2y = 0. \tag{28}$$

Conditions (24), (25), and (28) imply that $[s]$, for $s \in {}_2\Pi$, satisfies the relation used to define $\Gamma({}_2\Pi)$, while conditions (24), (25), and (26) imply that $[x, 0; h] = [0, y; h] = 0$ and that $[x, y; h] = -[y, x; h]$.

For the complex $A^2(\Pi)$, we have an isomorphism

$$H_4(A^2(\Pi)) \cong {}_2\Pi + \Lambda(\Pi),$$

where $\Lambda(\Pi)$ is the abelian group with generators $\langle x, y\rangle$ for all $x, y \in \Pi$, and relations

$$\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle; \qquad \langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle; \qquad \langle x, x\rangle = 0.$$

Finally, $H_4(A(\Pi)) = H_4(A^3(\Pi)) \cong {}_2\Pi$.

Closely connected with Theorem 3 is the fact that the symmetric three-dimensional cohomology group is zero, in the following sense.

THEOREM 4. *If the function* $f(x, y, z) \in G$ *on* Π *satisfies* (19) *and the symmetry condition*

$$f(x, y, z) - f(x, z, y) + f(z, x, y) = 0,$$

then there is a function $h(x, y) \in G$ *with* $h(x, y) - h(y, x) = 0$, *and*

$$f(x, y, z) = h(y, z) - h(x + y, z) + h(x, y + z) - h(x, y).$$

[3] This is the group $\Gamma(\Pi)$ used by J. H. C. Whitehead; see his article in these Proceedings.

In other words $\delta f = 0$ implies $f = \delta h$, when f and h both satisfy symmetry conditions, and δ is the coboundary operator of $A^0(\Pi)$.

For the case when Π is the additive group of integers the homology groups of $A(\Pi)$ can be expressed by means of direct sums of cyclic groups (m) of order m, as follows:

$$H_2 = H_4 = H_6 = H_8 = 0, \qquad H_{10} = (2)$$

$$H_1 = (\infty), \qquad H_3 = (2), \qquad H_5 = (2) + (3)$$

$$H_7 = (2) + (2), \qquad H_9 = (2) + (2) + (3) + (5), \qquad H_{11} = (2) + (2).$$

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