TOPOLOGICAL ALGEBRA[1]

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Topological algebra made its first appearance in the paper of Kürschak [9], where the definition of an abstract field with a valuation is clearly set forth. The foundation was completed in the thesis of van Dantzig [3]; topological groups, rings, fields, and linear spaces are there defined, and their basic properties are established.

In 1933 an important advance occurred when Haar [7] demonstrated the existence of an invariant measure in any locally compact group. Since then, our understanding of the structure of locally compact groups has rapidly grown. Most notably, Hilbert's fifth problem has been solved for compact groups and for solvable groups, and the general case appears to be within reach.

With the appearance of Banach's book [1] in 1932, the work of the Polish school on functional spaces reached its climax. With the perspective of two more decades, it seems to be fair to say that the expectation that Banach spaces would be a definitive unifying concept has been partly disappointed. Banach spaces are at once too general and too special; they are too general in that our detailed knowledge of Hilbert space shows no prospect of being extended to Banach spaces. They are too special in that they do not cover important instances of topological linear spaces that are met in applications.

Somewhat crudely, we may describe the present situation in the theory of topological linear spaces as follows. Banach's results fall into two types. In the first type the arguments are of a highly algebraic nature; here a great clarification was achieved by Mackey's program [10] of studing pairs of dual linear spaces in a purely algebraic way. In the second type of theorem, an important part of the proof rests on a topological device, or above all on a category argument. Here considerable progress has been made recently by Dieudonné and Schwartz [2]; they investigate topological linear spaces admitting a complete metric and direct limits of such spaces. The results have applications in Schwartz's theory of distributions.

In the present decade, much work has been devoted to Banach algebras (normed rings) since the publication of Gelfand's paper [5]. Gelfand attracted attention by his observation that a useful theorem due to Wiener could be regarded as a statement about inverses and maximal ideals in a suitable Banach algebra. Segal [12] and independently Godement [6] have pushed this line of investigation further.

In the study of Banach algebras themselves, there are at present two directions in which work is being done. One may attempt to probe more deeply

[1] This is a brief summary of the oral report delivered by the spokesman, who bears sole responsibility for it. A full account, prepared jointly by all three members of the panel, will appear elsewhere.

into the structure of a fairly general commutative Banach algebra; Silov, in a series of papers of which [14] is typical, has made some noteworthy contributions to this program. In the noncommutative case, study is at present largely confined to self-adjoint uniformly closed algebras of operators on Hilbert space (called C*-algebras by Segal in [13]). A commutative C*-algebra (say with unit) is completely known: it admits a faithful representation as the algebra of all continuous complex functions on a compact Hausdorff space. In a mild generalization of the commutative case, Kaplansky [8] finds that a suitably weakened version of this theorem is still valid.

Concerning the structure of the most general C*-algebra, there is as yet little that can be said as long as we merely assume uniform closure. If however we strengthen this hypothesis to weak closure, then a fairly definitive structure theory becomes available; it was given by Murray and von Neumann in a series of five memoirs, beginning with [11]. The theory of weakly closed algebras has again been taken up recently by Dixmier [4] and others, and there is every indication that our knowledge of them will be considerably increased.

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